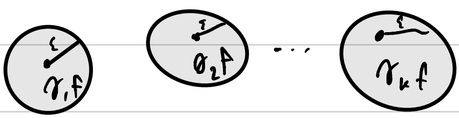
Ergodic Theory and Measured Group Theory
Lecture 16

Compact systems. Let Γ be a ctbl semigroup, e.g. $\mathbb{N}$, $\mathbb{Z}$, any other group. Let $\Gamma \curvearrowright (X,\mu)$ be a pmp action on (X,μ), i.e. each $\gamma \in \Gamma$ acts as a pmp transformation. This induces an action by isometries on $L^2(X,\mu)$ by $\gamma \cdot f := f \circ \gamma$.

$$\left(\|\gamma \cdot f\|_2 = \left(\int |f(\gamma(x))|^2 d\mu(x)\right)^{\frac{1}{2}} \overset{\text{pmp}}{=} \left(\int |f|^2 d\mu\right)^{\frac{1}{2}} = \|f\|_2 . \right)$$

Def. Call $\Gamma \curvearrowright (X,\mu)$ *compact* if every $f \in L^2(X,\mu)$ is *almost periodic*, i.e. $\Gamma \cdot f := \{\gamma \cdot f : \gamma \in \Gamma\}$ is precompact $:\Leftrightarrow \overline{\Gamma \cdot f}$ is compact $\Leftrightarrow \forall \varepsilon > 0\ \exists$ a finite ε-net in $\Gamma \cdot f$, i.e. $\exists\ \gamma_1, \dots, \gamma_k$ s.t. $\forall \gamma \in \Gamma$, $\|\gamma \cdot f - \gamma_i \cdot f\|_2 < \varepsilon$ for some $i = 1, \dots, k$.

$\Gamma \cdot f$:



Examples.

- Any rotation of $\mathbb{T} := \mathbb{R}/\mathbb{Z} = S^1$ by angle α is compact.

  Proof. This action is by taking $\alpha \in \mathbb{R}/\mathbb{Z}$ and acting by left addition by α, i.e. $\alpha \cdot x := \alpha + x$, $\forall x \in \mathbb{R}/\mathbb{Z}$. Thus $\forall f \in L^2(\mathbb{R}/\mathbb{Z}, \text{Lebesgue})$,

$$\overline{\{n\alpha\cdot f : n\in\mathbb{Z}\}}^{L^2} = \overline{\{n\alpha : n\in\mathbb{Z}\}}^{\mathbb{R}/\mathbb{Z}}\cdot f$$

hence $\overline{\{n\alpha : n\in\mathbb{Z}\}}$ is compact, hence closed and the map $n\alpha \mapsto n\alpha\cdot f$ is continuous, so $\overline{\{n\alpha : n\in\mathbb{Z}\}}\cdot f$ is a continuous image of the compact set $\overline{\{n\alpha : n\in\mathbb{Z}\}}$, thus compact.

- Let G be any compact group and $\Gamma \leq G$ a subgroup. Let μ denote the normalized Haar measure on G. Then $\Gamma \curvearrowright G$ be left multip. and this preserves μ, so pump. $\forall f\in L^2(G,\mu)$, $\overline{\Gamma\cdot f} = \overline{\Gamma}\cdot f$ which is compact being a continuous image of the compact set $\overline{\Gamma}$. So $\Gamma \curvearrowright G$ is a compact action. This is called a Kronecker action.

- Theorem. Any ergodic compact action of $\mathbb{Z}$ is a Kronecker action, i.e. $\exists$ compact group G (may take abelian) and $g\in G$ s.t. $\langle g\rangle \curvearrowright G$ is measure-isomorphic to the original action.

Counter-examples. All weakly mixing actions are "anti" compact.

Def. For a semigroup Γ, a set $S \subseteq \Gamma$ is called *syndetic* if $\exists\ \gamma_1, \gamma_2, \dots, \gamma_k \in \Gamma$ s.t. $\Gamma = \bigcup_{i=1}^{k} \gamma_i \cdot \Gamma$.

Example. $\Gamma := \mathbb{N}$ or $\mathbb{Z}$. Then $S \subseteq \Gamma$ is syndetic $\iff$ it it has bounded gaps (and includes 0 if $\Gamma = \mathbb{N}$).

Prop. A pmp action $\Gamma \curvearrowright (X, \mu)$ of a ctbl semigroup is compact $\iff$ $\forall f \in L^2(X, \mu)$ $\forall \varepsilon > 0$, the set $\{\gamma \in \Gamma : \|\gamma \cdot f - f\|_2 < \varepsilon\}$ is syndetic.

Proof. Homework.

Furstenberg Multiple Recurrence for compact $\mathbb{Z}$-actions. Let $\mathbb{Z} \overset{T}{\curvearrowright} (X, \mu)$ be a compact pmp action. Then $\forall A \subseteq X$ of positive measure and any $k \geq 0$ $\exists n \geq 1$ s.t.

$$\mu\left(A \cap T^{-n} A \cap T^{-2n} A \cap \dots \cap T^{-kn} A\right) > 0.$$

In fact, $\forall f \in L^\infty(X, \mu)$, $f \geq 0$, $\int f > 0$, $\forall k$,

$$\liminf_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} \int f \cdot (T^n f)(T^{2n} f) \dots (T^{kn} f)\, d\mu > 0.$$

Proof. Homework.

Szemerédi's theorem. In 1927, van der Waerden proved that if we partition $\mathbb{N}$ into finitely-many pieces then one of the pieces has arbitrarily long arithmetic progressions. In 1936, Erdős and Turán conjectured that this should be true for any subset of $\mathbb{N}$ of positive upper density.

Theorem (Szemerédi, 1975). This is true: every subset $A \subseteq \mathbb{N}$ ($\mathbb{Z}$) of positive $\bar{d}$ contains arbitrarily large arithmetic progressions. In fact, $\forall k \ \exists n$

$$\bar{d}(A \cap (A-n) \cap (A-2n) \cap \dots \cap (A-kn)) > 0.$$

In 1977, Furstenberg gave a different, ergodic theoretic, proof of this, which started a subject called ergodic Ramsey theory. What Furstenberg proved is this:

Furstenberg Multiple Recurrence Theorem (1977). For any pmp $\mathbb{Z} \curvearrowright (X, \mu)$, any $A \subseteq X$ of positive measure, $\forall k \ \exists n$ s.t.

$$\mu(A \cap T^{-n} A \cap \dots \cap T^{-kn}) > 0.$$

In fact, $\forall f \in L^\infty(X,\mu)$, $\forall k$, $f \geq 0$, $\int f \, d\mu > 0$:

$$\liminf_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} \int f \cdot (T^n f) \cdot \ldots \cdot (T^{kn} f) \, d\mu > 0.$$

Then Furstenberg showed that the Multiple Recurrence implies Szemerédi's theorem via a correspondence principle.

Furstenberg Correspondence Principle. For $\mathbb{N}$. For any $A \subseteq \mathbb{N}$ $\exists$ a pmp action $\mathbb{N} \overset{T}{\curvearrowright} (X,\mu)$ and $\tilde{A} \subseteq X$ s.t. $\mu(\tilde{A}) = \bar{d}(A)$ and for any $n_1, n_2, \dots, n_k \in \mathbb{N}$,

$$\bar{d}\left(A \cap (A-n_1) \cap (A-n_2) \cap \dots \cap (A-n_k)\right) \geq \mu\left(\tilde{A} \cap T^{-n_1}\tilde{A} \cap \dots \cap T^{-n_k}\tilde{A}\right).$$

For any amenable semigroup Γ. For any $A \subseteq \Gamma$ $\exists$ a pmp action $\Gamma \curvearrowright (X,\mu)$ and $\tilde{A} \subseteq X$ s.t. $\bar{d}(A) = \mu(\tilde{A})$ and $\forall\, g_1, g_2, \dots, g_k \in \Gamma$

$$\bar{d}\left(A \cap g_1^{-1}A \cap \dots \cap g_k^{-1}A\right) \geq \mu\left(\tilde{A} \cap g_1^{-1}\tilde{A} \cap \dots \cap g_k^{-1}\tilde{A}\right).$$

Here, by $\bar{d}$ we mean fix a Følner sequence (F_n) and

$$\bar{d}(A) := \limsup_{n\to\infty} \frac{|A \cap F_n|}{|F_n|}.$$

Proof. Consider the shift action $\Gamma \curvearrowright 2^\Gamma$ by $(\gamma\cdot x)(\delta) := x(\delta\cdot\gamma)$.

By passing to a subsequence, we may assume $\bar{d}(A) = \lim_{n\to\infty} \frac{|A\cap F_n|}{|F_n|}$.

Let $\tilde{A} := \{x \in 2^\Gamma : x(e_\Gamma) = 1\}$.

<u>Claim.</u> $\forall\, \gamma, g \in \Gamma$, $\gamma\cdot \mathbb{1}_A \in g^{-1}\tilde{A} \iff \gamma \in g^{-1}A$.

Proof. $\gamma\cdot\mathbb{1}_A \in g^{-1}\tilde{A} \iff g\gamma\cdot\mathbb{1}_A \in \tilde{A} \iff g\gamma\cdot\mathbb{1}_A(e_\Gamma) = 1$

$$\Leftrightarrow \mathbb{1}_A(g\gamma) = 1 \Leftrightarrow g\gamma \in A \Leftrightarrow \gamma \in g^{-1}A. \qquad \boxtimes$$

$$\mu_n := \frac{1}{|F_n|} \sum_{\gamma \in F_n} \delta_{\gamma \cdot \mathbb{1}_A}.$$